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WELL-POSEDNESS OF INITIAL VALUE PROBLEM FOR DISCRETE NONLINEAR WAVE EQUATIONS

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Abstract

We consider the initial value problem for discrete nonlinear wave equations. Under natural assumptions, we prove results on global well-posedness in a wide class of weighted l^2 spaces. Admissible spaces include spaces power and exponential decaying sequences.

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1 Introduction

In this paper we consider discrete nonlinear wave equations of the form

$$\ddot{q}_n = a_n q_{n+1} + a_{n-1} q_{n-1} + b_n q_n - f_n(q_n)\,, \quad n \in \mathbb{Z}\,, \tag{1.1}$$

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where the coefficients a_n and b_n are sequences of real numbers, and the nonlinearity f_n is a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(0) = 0$. Here and in what follows $\dot{}$ and $\ddot{}$ stand for the first and second time derivatives respectively. The unknown $q_n(t)$ is a sequence of real functions of real variable t. We study the initial value problem for equation (1.1) with initial conditions

$$q_n(0) = q_n^{(0)}, \quad \dot{q}_n(0) = q_n^{(1)}, \quad n \in \mathbb{Z}, \tag{1.2}$$

where $q_n^{(0)}$ and $q_n^{(1)}$ are given real sequences.

In fact, (1.1) is an infinite sequence of ordinary differential equations. But a better point of view is to consider equation (1.1) as an operator differential equation

$$\ddot{q} = Aq - B(q) \tag{1.3}$$

in certain Hilbert, or even Banach, space E of sequences. Here A is the linear operator defined by

$$(Aq)_n = a_n q_{n+1} + a_{n-1} q_{n-1} + b_n q_n, \quad n \in \mathbb{Z}, \tag{1.4}$$

and B is the nonlinear operator defined by

$$(B(q))_n = f_n(q_n), \quad n \in \mathbb{Z}. \tag{1.5}$$

Within this framework, initial conditions (1.2) become

$$q(0) = q^{(0)}, \quad \dot{q}(0) = q^{(1)}, \tag{1.6}$$

where $q^{(0)}$ and $q^{(1)}$ are given elements of the space E.

The simplest choice of such space is $E = l^2$, the space of two-sided square summable sequences. In this space equation (1.1) is Hamiltonian. In [4] (see also [9, Section 1.4]) the Hamiltonian structure, together with the classical existence and uniqueness theorem for operator differential equations and a cut-off argument, is used to obtain rather general global well-posedness of the initial value problem in l^2. We review those results in Section 2. The aim of the present paper is to extend the l^2-well-posedness results to weighted l^2-spaces and, hence, provide a refined information about problem (1.1), (1.2). This is done in Section 4. Similar idea has been used in [7] to study the discrete nonlinear Schrödinger equation. In Section 3 we discuss weighted l^2-spaces l^2_Θ and operators in such spaces. Section 5 is devoted to simplest examples appearing in applications.

2 Hamiltonian Structure and l^2-theory

Throughout the paper we impose the following assumptions.

(i) *The coefficients a_n and b_n are bounded real sequences .*

(ii) *The nonlinearity f_n is a real valued function on $\mathbb{R}$ such that $f_n(0) = 0$, and f_n is locally Lipschitz continuous uniformly with respect to $n \in \mathbb{Z}$,* i.e., *for any $R > 0$ there exists a constant $C(R) > 0$ such that*

$$|f_n(r_1) - f_n(r_2)| \le C(R)|r_1 - r_2|, \quad |r_1|, |r_2| \le R, \quad n \in \mathbb{Z}.$$

Sometimes we use the following stronger than (ii) assumption

(ii') *Assumption* (ii) *is satisfied with the constant* C *independent of* R, i.e., *there exists a constant* $C > 0$ *such that*

$$|f_n(r_1) - f_n(r_2)| \le C|r_1 - r_2|, \quad n \in \mathbb{Z}.$$

We denote by l^2 the Hilbert space of two-sided square summable sequences. The norm and inner product in this space are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Occasionally, we shall use more general spaces l^p, $1 \le p \le \infty$. The space l^p, $1 \le p < \infty$, consists of two-sided real sequences $u = (u_n)$ such that the norm

$$\|u\|_{l^p} = (\sum_{n\in\mathbb{Z}} |u_n|^p)^{1/p}$$

is finite. The space l^∞ consists of all bounded sequences. The norm in this space is given by

$$\|u\|_{l^\infty} = \sup_{n\in\mathbb{Z}} |u_n|.$$

Assumption (i) guaranties that the operator A is a bounded self-adjoint operator in l^2. With this choice of the configuration space, the phase space of equation (1.1) is $l^2 \times l^2$, and the equation is a Hamiltonian system. The Hamiltonian is given by

$$H(q,p) = \frac{1}{2}[\|p\|^2 - (Aq,q)] + \sum_{n=-\infty}^{\infty} F_n(q_n),$$

where

$$F_n(r) = \int_0^r f_n(s)\,ds$$

is the primitive function of f_n. The Hamiltonian H is a C^1 functional on the phase space and, hence a conserved quantity, *i.e.*, for any solution of equation (1.1) or, equivalently, (1.3)

$$H(q,\dot{q}) = \text{const}.$$

Now we reproduce some results from [4] (see also [9, Section 1.4]). The first one is a simple straightforward consequence of classical theorems on existence and uniqueness of global solutions for operator differential equations (see, *e.g.*, [6, Chaptrer 6, Theorem 1.2] and [10, Chapter 6, Theorems 1.2 and 1.4]). This result does not use the Hamiltonian structure of equation (1.1).

Theorem 2.1. *Under assumptions* (i) *and* (ii'), *for every* $q^{(0)} \in l^2$ *and* $q^{(1)} \in l^2$ *there exists a unique solution* $q \in C^2(\mathbb{R}, l^2)$ *of problem (1.1), (1.2).*

The proof of the next theorem makes use of Theorem 2.1, the Hamiltonian structure of the equation and a cut-off argument.

Theorem 2.2. *Assume* (i) *and* (ii). *Suppose that the operator* A *is non-positive,* i.e., $(Aq,q) \le 0$ *for all* $q \in l^2$ *and* $F_n(r) \ge 0$ *for all* $r \in \mathbb{R}$. *Then problem (1.1), (1.2) has a unique global solution* $q \in C^2(\mathbb{R}, l^2)$ *for all* $q^{(0)} \in l^2$ *and* $q^{(1)} \in l^2$.

A completely different type of nonlinearities is considered in the following

Theorem 2.3. *Assume (i), and let $f_n(r) =$ be a positively homogeneous function of degree $p > 1$ such that $|f_n(\pm 1)| \le C$ for some positive constant C. Suppose that the operator A is negative definite,* i.e.,

$$(Aq, q) \le -\alpha \|q\|^2 \,, \tag{2.1}$$

where $\alpha > 0$. Then there exists $\delta > 0$ such that for every $q^{(0)} \in l^2$ and $q^{(1)} \in l^2$, with $\|q^{(0)}\| < \delta$ and $\|q^{(1)}\| < \delta$, problem (1.1), (1.2) has a unique solution $q \in C^2(\mathbb{R}, l^2)$. The solution q is a bounded function with values in l^2.

Let us point out that in [4] Theorem 2.3 is proven in the case when $f_n(r) = d_n r^2$. The general case requires only minor changes in the proof.

Now we supplement Theorem 2.2 with the following result on the boundedness of the solution.

Theorem 2.4. *Assume that (i) and (ii) are satisfied, and $F_n(r) \ge 0$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{R}$.*

(a) If the operator A is non-positive and $\lim_{r\to\pm\infty} F_n(r) = +\infty$ uniformly with respect to $n \in \mathbb{Z}$, then the unique solution of problem (1.1), (1.2), with $q^{(0)} \in l^2$ and $q^{(1)} \in l^2$, is a bounded function on $\mathbb{R}$ with values in l^∞. In addition, if, for some $s \ge 2$, there exist $R > 0$ and $c > 0$ such that

$$F_n(r) \ge c|r|^s \,, \quad \forall r \in [-R, R] \,, \forall n \in \mathbb{Z} \,, \tag{2.2}$$

then the solution is a bounded function on $\mathbb{R}$ with values in l^s.

(b) If the operator A is negative definite, then the unique solution of problem (1.1), (1.2), with $q^{(0)} \in l^2$ and $q^{(1)} \in l^2$, is a bounded function on $\mathbb{R}$ with values in l^2.

Proof. (a) We have that

$$H(q(t), \dot q(t)) = \frac{1}{2}[\|\dot q(t)\|^2 - (Aq(t), q(t))] + \sum_{n=-\infty}^{\infty} F_n(q_n(t)) = H(q^{(0)}, q^{(1)}) \tag{2.3}$$

because the Hamiltonian H is a conserved quantity. Since A is non-positive while F_n is non-negative, this implies that

$$F_n(q_n(t)) \le H(q^{(0)}, q^{(1)}) \,.$$

Therefore, there exists a constant $C > 0$ such that $|q_n(t)| \le C$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ because F_n has infinite limit at infinity uniformly with respect to $n \in \mathbb{Z}$.

Let us prove the second part of statement (a). The assumption on the limit of F_n at infinity implies that if inequality (2.2) holds for some $R > 0$, then it holds for every $R > 0$, with the constant $c > 0$ depending on R. By the first part of the statement, there exists $R > 0$ such that $\|q(t)\|_{l^\infty} \le R$ for all $t \in \mathbb{R}$. Hence, by (2.3) and (2.2),

$$c \sum_{n=-\infty}^{\infty} |q_n(t)|^s \le H(q^{(0)}, q^{(1)})$$

for all $t \in \mathbb{R}$ which implies the required.

(b) In this case equation (2.3) and inequality (2.1) imply that

$$\alpha \|q(t)\|^2 \le H(q^{(0)}, q^{(1)})$$

for all $t \in \mathbb{R}$ and the result follows.

□

3 Weighted Spaces

Let $\Theta = (\theta_n)$ be a sequence of positive numbers (weight). The space l^2_Θ consists of all two-sided sequences of real numbers such that the norm

$$\|u\|_\Theta = (\sum_{n\in\mathbb{Z}} u_n^2\theta_n)^{1/2}$$

is finite. This is a Hilbert space.

We always suppose that the weight Θ satisfies the following regularity assumption:

(iii) *the sequence* Θ *is bounded below by a positive constant and there exists a constant* $c_0 > 0$ *such that*

$$c_0^{-1} \le \frac{\theta_{n+1}}{\theta_n} \le c_0$$

for all $n \in \mathbb{Z}$.

A weight satisfying assumption (iii) is called *regular*.

Obviously, under this assumption l^2_Θ is densely and continuously embedded into l^2 and

$$\|u\| \le C\|u\|_\Theta\,, \quad u \in l^2_\Theta\,,$$

with some $C > 0$. Therefore, all these spaces are densely and continuously embedded into the the space l^∞ of bounded sequences, with sup-norm. If $\theta_n \equiv 1$, then $l^2_\Theta = l^2$.

From the point of view of functional analysis assumption (iii) is quite natural. It means that the space l^2_Θ is translation invariant. More precisely, let T_+ and T_- be the operators of right and left shifts, respectively, defined by

$$(T_+w)_n = w_{n-1} \quad \text{and} \quad (T_-w)_n = w_{n+1}\,.$$

Lemma 3.1. *Assumption* (iii) *holds if and only if both* T_+ *and* T_- *are linear bounded operators in* l^2_Θ.

Proof. Indeed, we have that

$$\|T_+w\|^2_\Theta = \sum_{n\in\mathbb{Z}} w_{n-1}^2\theta_n = \sum_{n\in\mathbb{Z}} w_n^2\theta_n\frac{\theta_{n+1}}{\theta_n}\,.$$

Hence, T_+ is bounded in l^2_Θ if and only if θ_{n+1}/θ_n is bounded. Similarly, T_- is bounded in l^2_Θ if and only if θ_{n-1}/θ_n is bounded.

□

Note that T_+ and T_- are mutually inverse operators. But let us point out that the translation invariance of the space l^2_Θ does not mean that the norm $\|\cdot\|_{l^2_\Theta}$ is translation invariant.

The most important examples of regular weights are

(a) power weight

$$\theta_n = (1+|n|)^b\,, \quad b > 0\,; \tag{3.1}$$

(b) exponential weight

$$\theta_n = \exp(\alpha|n|)\,, \quad \alpha > 0\,. \tag{3.2}$$

More generally, the weight $\theta_n = \exp(\alpha|n|^\beta)$, $\alpha > 0$, satisfies assumption (iii) if and only if $0 < \beta \le 1$.

4 Well-posedness in Weighted Spaces

We start with two simple lemmas.

Lemma 4.1. *Assume (i). Let Θ be a regular weight. Then the operator A defined by equation (1.4) acts in l^2_Θ as a bounded linear operator.*

Proof. The operator A can be represented in the form

$$A = a \circ T_- + T_+ \circ a + b\,,$$

where a and b are the operators of multiplication by the sequences (a_n) and (b_n) respectively, and $\circ$ stands for the composition of operators. The operators T_-, T_+, a and b are bounded operators in l^2_Θ by Lemma 3.1 and assumption (i) respectively. Hence, the result follows.

□

Lemma 4.2. *Under assumption (ii), the nonlinear operator B defined by equation (1.5) is a locally Lipschitz continuous operator in the space l^2_Θ,* i.e., *for any $R > 0$ there exists a constant $C_R > 0$ such that*

$$\|B(v) - B(w)\|_{l^2_\Theta} \le C_R \|v - w\|_{l^2_\Theta} \tag{4.1}$$

for all $v \in l^2_\Theta$ and $w \in l^2_\Theta$ such that $\|v\|_{l^2_\Theta} \le R$ and $\|w\|_{l^2_\Theta} \le R$. If assumption (ii') is satisfied, then the operator B is Lipschitz continuous, i.e., *the constant in inequality (4.1) can be chosen independent of R.*

Proof. Straightforward.

□

Our key observation is the following

Theorem 4.3. *Assume (i), (ii) and (iii). Suppose that $q \in C^2((-T,T);l^2)$ is a solution of problem (1.1), (1.2) with $q^{(0)} \in l^2_\Theta$ and $q^{(1)} \in l^2_\Theta$. Then $q \in C^2((-T,T);l^2_\Theta)$.*

Proof. Let $q \in C^2((-T,T),l^2)$ be a solution of problem 1.1), (1.2) with $q^{(0)} \in l^2_\Theta$ and $q^{(1)} \in l^2_\Theta$. Pick any $\tau \in (0,T)$ and set $R_\tau = \sup_{t\in[-\tau,\tau]} \|u(t)\|$. Let $\tilde{f}_n(r) = f_n(r)$ if $|r| \le R_\tau + 1$ and $\tilde{f}_n(r) = f_n(R_\tau + 1)$ if $|r| > R_\tau + 1$. Then on $[-\tau,\tau]$ the function $q(t)$ obviously solves the equation

$$\ddot{q}_n = a_n q_{n+1} + a_{n-1} q_{n-1} + b_n q_n - \tilde{f}_n(q_n)\,, \quad n \in \mathbb{Z}\,, \tag{4.2}$$

with the same initial data.

Obviously, the functions $\tilde{f}_n$ satisfy assumption (ii'), and, by Lemma 4.2, the corresponding operator $\tilde{B}$ is globally Lipschitz continuous in the space l^2_Θ. By Lemma 4.1, the operator A is a bounded linear operator in l^2_Θ. By the classical result [6, Chaptrer 6, Theorem 1.2] and [10, Chapter 6, Theorems 1.2 and 1.4], problem (4.2), (1.2) has a unique solution $\tilde{q} \in C^2(\mathbb{R}, l^2_\Theta) \subset C^2(\mathbb{R}, l^2)$. By uniqueness for the initial value problem in the space l^2, we have that $\tilde{q} = q$ on $[-\tau,\tau]$. Since $\tau \in (0,T)$ is an arbitrary point, we obtain that $q \in C^2((-T,T), l^2_\Theta)$.

□

Combining Theorem 4.3 with Theorems 2.1 – 2.3, we obtain the following corollaries.

Corollary 4.4. *Under assumptions* (i) *and* (ii')*, for every* $q^{(0)} \in l^2_\Theta$ *and* $q^{(1)} \in l^2_\Theta$ *there exists a unique solution* $q \in C^2(\mathbb{R}, l^2_\Theta)$ *of problem (1.1), (1.2).*

Corollary 4.5. *Assume* (i) *and* (ii)*. Suppose that the operator A is non-positive,* i.e., $(Aq, q) \le 0$ *for all* $q \in l^2$ *and* $F_n(r) \ge 0$ *for all* $r \in \mathbb{R}$*. Then problem (1.1), (1.2) has a unique global solution* $q \in C^2(\mathbb{R}, l^2_\Theta)$ *for all* $q^{(0)} \in l^2_\Theta$ *and* $q^{(1)} \in l^2_\Theta$*.*

Corollary 4.6. *Assume* (i)*, and let* $f_n(r)$ *be a positively homogeneous function of degree* $p > 1$ *such that* $|f_n(\pm 1)| \le C$ *for some positive constant C. Suppose that the operator A is negative definite,* i.e.,

$$(Aq, q) \le -\alpha \|q\|^2 , \tag{4.3}$$

where $\alpha > 0$*. Then there exists* $\delta > 0$ *such that for every* $q^{(0)} \in l^2_\Theta$ *and* $q^{(1)} \in l^2_\Theta$*, with* $\|q^{(0)}\| < \delta$ *and* $\|q^{(1)}\| < \delta$*, problem (1.1), (1.2) has a unique solution* $q \in C^2(\mathbb{R}, l^2_\Theta)$.

Let us highlight that in Corollary 4.6 the smallness of the initial data is with respect to the l^2-norm, not in the space l^2_Θ.

5 Examples

Now we present some examples that often appear in applications (see, *e.g.*, [1, 5, 11]). In these examples Δ stands for the one-dimensional Laplacian defined by

$$(\Delta q)_n = q_{n+1} + q_{n-1} - 2q_n .$$

The first example is the well-known *Frekel-Kontorova* (FK) model. The equation reads

$$\ddot{q}_n = a(\Delta q)_n - \sin q_n , \tag{5.1}$$

where $a > 0$. This is a straightforward discretization of the sin-Gordon equation

$$u_{tt} - a u_{xx} + \sin u .$$

The last equation is a completely integrable system (see, *e.g.*, [2]). At the same time its discrete counterpart (5.1) is *not* completely integrable.

In the case of equation (5.1) the nonlinearity satisfies (ii'). Hence, Corollary 4.4 shows that the initial value problem for (5.1) is globally well-posed in every space l^2_Θ with a regular weight Θ.

Now consider the equation

$$\ddot{q}_n = a(\Delta q)_n - m^2 q_n \pm q_n^3 . \tag{5.2}$$

If the sign in front of the cubic nonlinearity is positive, this is the *repulsive discrete nonlinear Klein-Gordon* (DNKG_-) equation in case when $m^2 > 0$, and *repulsive discrete nonlinear wave* (DNW_-) equation when $m^2 = 0$. In case of negative sign, we obtain the *attractive discrete nonlinear Klein-Gordon* (DNKG_+) equation ($m^2 > 0$) and the *attractive discrete nonlinear wave* (DNW_+) equation ($m^2 = 0$) respectively.

It is easy to verify that

$$(\Delta q, q) = \sum_{n\in\mathbb{Z}} (q_n - q_{n-1})^2$$

and, hence, the operator Δ is nonnegative. By Corollary 4.5, in the attractive case the initial value problem for both DNKG$_+$ and DNW$_+$ is globally well-posed in all spaces l^2_Θ with regular weight Θ. This is because $F_n(r) = r^4/4 \geq 0$. On the other hand, in the repulsive case $F_n(r) = -r^4/4 \leq 0$. In case of DNKG$_-$ the operator $\Delta - m^2$ is negative definite, and Corollary 4.6 guaranties the existence of global solution in l^2_Θ for all initial data in l^2_Θ that have sufficiently small l^2-norm, provided the weight Θ is regular. The case of DNW$_-$ remains open.

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