SECTION 1. MODERN PROBLEMS OF MATHEMATICS

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ON EXISTENCE OF SOLITARY TRAVELING WAVES IN FERMI-PASTA-ULAM TYPE SYSTEMS ON 2D-LATTICE

Abstract. We consider the Fermi-Pasta-Ulam type systems with saturable nonlinearities that describes an infinite systems of particles on a two dimensional lattice. The main result concerns the existence of solitary traveling waves solutions with vanishing relative displacement profiles. By means of critical point theory, we obtain sufficient conditions for the existence of such solutions.

Key words and phrases: traveling waves, Fermi-Pasta-Ulam type systems, 2D-lattice, saturable nonlinearities, critical point theory.

Recently, considerable attention has been paid to models that are discrete in the spatial variables. Among the equations that describe such models, the most famous are the Discrete Nonlinear Shrödinger type equations, the Discrete Sine-Gordon type equations, the equations of chains of oscillators and the Fermi-Pasta-Ulam type systems. Such equations are of interest in view of numerous applications in physics. Among the solutions of such systems, traveling waves deserve special attention.

We study the Fermi-Pasta-Ulam type systems that describes the dynamics of an infinite systems of nonlinearly coupled particles on a two dimensional lattice. Let $q_{n,m} = q_{n,m}(t)$ be a generalized coordinate of the (n,m)-th particle at time t. It is assumed that each particle interacts nonlinearly with its four nearest neighbors. The equations of motion of the system considered are of the form

$$\begin{aligned} \ddot{q}_{n,m}(t) = W_1'(q_{n+1,m}(t) - q_{n,m}(t)) - W_1'(q_{n,m}(t) - q_{n-1,m}(t)) + \\ +W_2'(q_{n,m+1}(t) - q_{n,m}(t)) - W_2'(q_{n,m}(t) - q_{n,m-1}(t)), (n,m) \in \mathbb{Z}^2, \end{aligned} \tag{1}$$

where $W_1, W_2 \in C^1(\mathbb{R}; \mathbb{R})$ are interaction potentials.

In contrast to the previous results (see [2] and [3]), we study system (1) with saturable nonlinearities which means that at infinity $W_i'(r)$ growth as $const \cdot r$, i.e. $W_i(r)$ are asymptotically quadratic at infinity $(i = 1,2)$. Note that in [4] and [6] such nonlinearities are considered.

A traveling wave solution of Eq. (1) is a function of the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct), \tag{2}$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$\begin{aligned} c^2u''(s) = W_1'(u(s+\cos\varphi) - u(s)) - W_1'(u(s) - u(s-\cos\varphi)) + \\ +W_2'(u(s+\sin\varphi) - u(s)) + W_2'(u(s) - u(s-\sin\varphi)), \end{aligned} \tag{3}$$

where $s = n\cos\varphi + n\sin\varphi - ct.$

In what follows, a solution of Eq. (3) is understood as a function $u(s)$ from the space $C^2(\mathbb{R};\mathbb{R})$ satisfying Eq. (3) for all $s \in \mathbb{R}$.

We consider two types of solutions: periodic and solitary traveling waves. In the first case profile satisfies the following periodicity condition

$$u'(s+2k) = u'(s), \tag{4}$$

where $k > 0$ is a real number. Note that the profile of such wave is not necessarily periodic. In the second case profile satisfies the following condition

$$\lim_{s\to\pm\infty} u'(s) = u'(\pm\infty) = 0, \tag{5}$$

i.e., the relative displacement profiles

$$r_1^{\pm}(s) = \int_s^{s\pm\cos\varphi} u'(\tau)d\tau, \quad r_2^{\pm}(s) = \int_s^{s\pm\sin\varphi} u'(\tau)d\tau$$

vanish at infinity.

We always assume that

(i) $W_i(r) = \frac{c_i^2}{2} r^2 + f_i(r)$, *where* $c_i \in\in \mathbb{R}$, $f_i \in C^1(\mathbb{R})$, $f_i(0) = f_i'(0) = 0$ *and*

$f_i'(r) = o(r)$ *as* $r \to 0$, $i = 1,2$;

(ii) *there exists a finite limit* $\lim\limits_{r\to\pm\infty} \frac{f_i'(r)}{r} = l$ *and the functions* $g_i(r) = f_i'(r) - l$ *are bounded* $(i = 1,2)$;

(iii) $f_i(r) \ge 0$ *forall* $r \in \mathbb{R}$ *and for every* $r_0 > 0$ *there exists* $\delta_0 = \delta_0(r_0) > 0$ *such that* $\frac{1}{2} r f_i'(r) - f_i(r) \ge \delta_0$ *for* $|r| \ge r_0$ $(i = 1,2)$.

The important role is played by the quantity defined by the equality

$$c_0 = c_0(\varphi) := \sqrt{c_1^2 \cos^2\varphi + c_2^2 \sin^2\varphi}.$$

Let E_k be the Hilbert space defined by

$$E_k = \{u \in H_{loc}^1(\mathbb{R}) : u'(s+2k) = u'(s),\ u(0) = 0\}$$

with the scalar product

$$(u,v) = \int_{-k}^{k} u'(s)v'(s)ds.$$

On E_k we introduce the functional

$$J_k(u) = \int_{-k}^{k} \left[\frac{c^2}{2}(u'(s))^2 - \frac{c_1^2}{2}(Au(s))^2 - \frac{c_2^2}{2}(Bu(s))^2 - f_1(Au(s)) - f_2 Bu(s)) \right] ds,$$

where

$$(Au)(s) := u(s+\cos\varphi) - u(s) = \int_{s}^{s+\cos\varphi} u'(\tau)d\tau,$$

$$(Bu)(s) := u(s+\sin\varphi) - u(s) = \int_{s}^{s+\sin\varphi} u'(\tau)d\tau.$$

The critical points of the functional J_k are solutions of Eq. (3) satisfying (4). Their existence is established in the paper [1].

Let E be the Hilbert space defined by

$$E = \{u \in H_{loc}^1(\mathbb{R}) : u' \in L^2(\mathbb{R}),\ u(0) = 0\}$$

with the scalar product

$$(u,v)=\int_{-\infty}^{+\infty} u'(s)v'(s)ds\,.$$

Note that the condition $u'\in L^2(\mathbb{R})$ in the definition of E corresponds to the condition (5) and the condition $u(0)=0$ is meaningful because every element of $H^1_{loc}(\mathbb{R})$ is a continuous function.

On E we introduce the functional

$$J(u)=\int_{-\infty}^{+\infty}\left[\frac{c^2}{2}(u'(s))^2-\frac{c_1^2}{2}(Au(s))^2-\frac{c_2^2}{2}(Bu(s))^2-f_1(Au(s))-f_2Bu(s))\right]ds.$$

Any critical point of the functional J is a solution of Eq. (3) satisfying (5). In a sense, the case of solitary waves is a limit case of the periodic waves. Therefore, solitary waves were constructed by considering critical points of the functional J_k and then passing to the limit as $k\to\infty$.

The main result is the following theorem.

Theorem 1 ([5]). *Assume* $(i)-(iii)$. *If* $\varphi\in\left[\pi n,\frac{\pi}{2}+\pi n\right]$, $n\in\mathbb{Z}$, *and* $c_0^2<c^2<c_0^2+l$, *then Eq. (3) has a non-constant non-decreasing and non-increasing solutions satisfying (5).*

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