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The Existence of Heteroclinic Travelling Waves in the Discrete Sine-Gordon Equation with Nonlinear Interaction on a 2D-Lattice

S. Bak

The article deals with the discrete sine-Gordon equation that describes an infinite system of nonlinearly coupled nonlinear oscillators on a 2D-lattice with the external potential $V(r) = K(1 - \cos r)$. The main result concerns the existence of heteroclinic travelling waves solutions. Sufficient conditions for the existence of these solutions are obtained by using the critical points method and concentration-compactness principle.

Key words: discrete sine-Gordon equation, nonlinear oscillators, 2D-lattice, heteroclinic travelling waves, critical points, concentration-compactness principle.

Mathematical Subject Classification 2010: 34G20, 37K60, 58E50.

1. Introduction

In the paper, we study the discrete sine-Gordon equation that describes the dynamics of an infinite system of nonlinearly coupled nonlinear oscillators on a two-dimensional lattice. Let $q_{n,m}$ be a generalized coordinate of the (n,m)-th oscillator at the time t. It is assumed that each oscillator interacts nonlinearly with its four nearest neighbors. The equation of motion of the system considered is of the form

$$\begin{aligned} \ddot{q}_{n,m} = V'(q_{n+1,m} - q_{n,m}) - V'(q_{n,m} - q_{n-1,m}) + V'(q_{n,m+1} - q_{n,m}) \\ - V'(q_{n,m} - q_{n,m-1}) - K\sin(q_{n,m}), \quad (n,m) \in \mathbb{Z}^2, \end{aligned} \tag{1}$$

where $K > 0$. Equations (1) form an infinite system of ordinary differential equations.

System (1) can be considered as a 2D version of the Frenkel–Kontorova model (see, e.g., [11]). Notice that this system represents a wide class of systems called lattice dynamical systems extensively studied in last decades. In this area of research, a great attention is paid to an important specific class of solutions called travelling waves solutions. A comprehensive presentation of the results on travelling waves for 1D Fermi–Pasta–Ulam lattices is given in [19]. The existence of periodic travelling waves in the Fermi–Pasta–Ulam system on a 2D-lattice is studied in [4]. On the other hand, some results on the chains of oscillators are also known in the literature. In particular, in [14] they are obtained by means of bifurcation theory, while in [1] and [2] the existence of periodic and solitary travelling waves is studied by means of the critical point theory. In papers [3,10,12,13], travelling waves for infinite systems of linearly coupled oscillators on a 2D-lattice are studied. Paper [18] is devoted to periodic and homoclinic travelling waves for the infinite one-dimensional chain of nonlinearly coupled nonlinear particles. In [6], a result on the existence of subsonic periodic travelling waves for the system of nonlinearly coupled nonlinear oscillators on a 2D-lattice is obtained, and in [7], supersonic periodic travelling waves for these systems are studied. Paper [15] contains a result on the existence of heteroclinic travelling waves for the discrete sine-Gordon equation with linear interaction. In [16], periodic, homoclinic and heteroclinic travelling waves for such systems with nonlinear interaction are studied. In paper [5], a result on the existence of periodic travelling waves for the discrete sine-Gordon equation with nonlinear interaction on a 2D-lattice is obtained. [8] is devoted to the existence of heteroclinic travelling waves for the discrete sine-Gordon equation with linear interaction on a 2D-lattice.

2. The problem statement

A travelling wave solution of equation (1) is a function of the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct)\,,$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$\begin{aligned} c^2u''(s) &= V'(u(s+\cos\varphi) - u(s)) - V'(u(s) - u(s-\cos\varphi)) \\ &\quad + V'(u(s+\sin\varphi) - u(s)) - V'(u(s) - u(s-\sin\varphi)) - K\sin(u(s)). \quad (2) \end{aligned}$$

The constant $c \neq 0$ is called the speed of the wave. If $c > 0$, then the wave moves to the right, otherwise to the left.

An important role is played by the quantity c_1 defined by the equation

$$c_1^2 := 2\sup_{|r|<6\pi}\left|\frac{V(r)}{r^2}\right|.$$

We consider the case of heteroclinic travelling waves. The profile function of this wave satisfies the conditions:

$$\lim_{s\to-\infty} u(s) = -\pi \quad \text{and} \quad \lim_{s\to+\infty} u(s) = \pi. \qquad (3)$$

In what follows, a solution of equation (2) is understood as a function $u(s)$ from the space $C^2(\mathbb{R})$ satisfying equation (2) for all $s \in \mathbb{R}$.

3. Variational setting

To equation (2), we associate the functional

$$J(u) := \int_{-\infty}^{+\infty} \Big[\frac{c^2}{2}(u'(s))^2 - V(u(s+\cos\varphi) - u(s)) \\ -V(u(s+\sin\varphi) - u(s)) + K(1+\cos(u(s))) \Big]\, ds, \tag{4}$$

defined on the Hilbert space

$$E := \{u \in H^1_{\mathrm{loc}}(\mathbb{R}) : u' \in L^2(\mathbb{R})\}$$

with the scalar product

$$(u,v)_E = u(0)v(0) + \int_{-\infty}^{+\infty} u'(s)v'(s)\, ds.$$

It is not so difficult to verify that the critical points of the functional J are the solutions of equation (2).

Now we introduce the following notation:

$$\begin{aligned} \mathcal{M}_{-\pi,\pi} &= \{u \in E : u(-\infty) = -\pi, u(+\infty) = \pi\}, \\ Au(s) &:= u(s+\cos\varphi) - u(s), \\ Bu(s) &:= u(s+\sin\varphi) - u(s). \end{aligned}$$

According to Lemma 3.1 from [10],

$$\begin{aligned} \|Au(s)\|_{L^2(\mathbb{R})} &\le |\cos\varphi| \cdot \|u'(s)\|_{L^2(\mathbb{R})}, \qquad && u \in E, \\ \|Bu(s)\|_{L^2(\mathbb{R})} &\le |\sin\varphi| \cdot \|u'(s)\|_{L^2(\mathbb{R})}, \qquad && u \in E. \end{aligned}$$

Then the functional J can be expressed in the form

$$J(u) := \int_{-\infty}^{+\infty} \Big[\frac{c^2}{2}(u'(s))^2 - V(Au(s)) - V(Bu(s)) + K(1+\cos(u(s))) \Big]\, ds. \tag{5}$$

Throughout the paper we will assume that the interaction potential $V(r)$ satisfies the following conditions:

(i) $V(r) \in C^1(\mathbb{R})$, $V(0) = 0$ and $V(r) \ge 0$ for all $r \in \mathbb{R}$;

(ii) $\lim_{r\to\pm\infty} V(r) = +\infty$;

(iii) there exists finite $\lim_{r\to 0} \left|\frac{V(r)}{r^2}\right|$;

(iv) the wave speed c satisfies $c^2 > c_1^2$.

The following lemma can be obtained by a straightforward calculation (see [15] for details).

Lemma 3.1. *Let $v_0 : \mathbb{R} \to [-\pi, \pi]$ be a monotone function in $C^\infty(\mathbb{R})$ such that $v_0(s) = -\pi$ for $s < -1$ and $v_0(s) = \pi$ for $s > 1$. Define the functional $\Psi : H^1(\mathbb{R}) \to \mathbb{R}$ by*

$$\Psi(v) := J(v_0 + v)$$

and suppose that assumptions (i)–(iv) are satisfied. Then the following holds:

(i_1) *$\Psi(v) < +\infty$ for all $v \in H^1(\mathbb{R})$ (equivalently, $J(u) < +\infty$ for all u of the form $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$);*

(ii_1) *$J(u) = +\infty$ for all $u \in \mathcal{M}_{-\pi,\pi}$ which are not of the form $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$. In particular, a minimizer u of J on $\mathcal{M}_{-\pi,\pi}$ can be expressed as $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$;*

(iii_1) *$\Psi \in C^1$ on $H^1(\mathbb{R})$;*

(iv_1) *let $v \in H^1(\mathbb{R})$ be a critical point of Ψ and set $u := v_0 + v$. Then $u, v \in C^2(\mathbb{R})$, and u is a solution of (2) with boundary conditions (3).*

Let F be a non-negative function in $C^\infty(\mathbb{R})$ such that

$$\begin{cases} F(r) = 0, & \text{if } |r| \le \frac{5\pi}{2}, \\ F(r) \ge 4\left|\int_0^{2r} |V'(x)|dx\right| \text{ and } F(r) \ge 2K, & \text{if } |r| \ge 3\pi, \\ \frac{1}{2} \le 1 + \cos r + \frac{1}{2K}F(r), & \text{if } |r| \in \left(\frac{5}{2}\pi, 3\pi\right). \end{cases} \tag{6}$$

Now we define the modified functional $\tilde{J} : E \to \mathbb{R} \cup \{\infty\}$ by

$$\tilde{J}(u) := \int_{-\infty}^{+\infty} \left[\frac{c^2}{2}(u'(s))^2 - V(Au(s)) - V(Bu(s)) \right. \\ \left. + K(1 + \cos(u(s))) + F(u(s))\right] ds. \tag{7}$$

Remark 3.2. Obviously, $\tilde{J}(u) = J(u)$ for all $u \in E$ with norm

$$\|u\|_{L^\infty(\mathbb{R})} \le \frac{5}{2}\pi.$$

Now we denote the modified potential of interaction by

$$\tilde{V}(r) = \left|\int_0^r |V'(x)|dx\right|.$$

Then from (6) for all $|r| \ge 3\pi$, we have

$$V(2r) \le \tilde{V}(2r) \le \frac{1}{4}F(r). \tag{8}$$

Hence, by (ii), $F(r) \to +\infty$ for $r \to \pm\infty$.

The lemma below can be found in [16, Lemma 2.5].

Lemma 3.3. *Let $W \in C^1(\mathbb{R})$ be such that $W(\pm\pi) = 0$ and $W(\xi) > 0$ for $|\xi| < \pi$, and let*

$$I(u) := \int_{-\infty}^{+\infty} [(u'(s))^2 + W(u(s))]ds.$$

Then the minimum of I on $\mathcal{M}_{-\pi,\pi}$ is attained and

$$\min_{u\in\mathcal{M}_{-\pi,\pi}} I(u) = 2\int_{-\pi}^{\pi} \sqrt{W(\xi)}\, d\xi =: \vartheta.$$

Moreover, with the same ϑ,

$$\inf_{T>0} \inf_{u\in H^1(-T,T)} \left\{ \int_{-T}^{T} [(u'(s))^2 + W(u(s))]\, ds : u(-T) = -\pi, u(T) = \pi \right\} = \vartheta.$$

Lemma 3.4. *Assume conditions (i)–(iv) hold. Then for all $u \in E$,*

$$\tilde{J}(u) \geq \int_{-\infty}^{+\infty} \left[\frac{c^2 - c_1^2}{2}(u'(s))^2 + K(1 + \cos(u(s)) + \frac{1}{2}F(u(s)) \right] ds, \tag{9}$$

and the functional $\tilde{J}$ is bounded from below on $\mathcal{M}_{-\pi,\pi}$. Moreover,

$$8\sqrt{(c^2 - c_1^2)K} < \inf_{u\in\mathcal{M}_{-\pi,\pi}} \tilde{J}(u) < 8c\sqrt{K}. \tag{10}$$

Proof. Since

$$|Au(s)| \leq |u(s + \cos\varphi)| + |u(s)| \leq 2\max\{|u(s + \cos\varphi)|, |u(s)|\},$$
$$|Bu(s)| \leq |u(s + \sin\varphi)| + |u(s)| \leq 2\max\{|u(s + \sin\varphi)|, |u(s)|\},$$

then for every $k > 0$,

$$\begin{aligned}
\{s \in \mathbb{R} : |Au(s)| > k\} &\subseteq \left\{ s \in \mathbb{R} : \max\{|u(s + \cos\varphi)|, |u(s)|\} > \frac{k}{2} \right\} \\
&\subseteq \left\{ s \in \mathbb{R} : |u(s + \cos\varphi)| > \frac{k}{2} \right\} \cup \left\{ s \in \mathbb{R} : |u(s)| > \frac{k}{2} \right\}, \\
\{s \in \mathbb{R} : |Bu(s)| > k\} &\subseteq \left\{ s \in \mathbb{R} : \max\{|u(s + \sin\varphi)|, |u(s)|\} > \frac{k}{2} \right\} \\
&\subseteq \left\{ s \in \mathbb{R} : |u(s + \sin\varphi)| > \frac{k}{2} \right\} \cup \left\{ s \in \mathbb{R} : |u(s)| > \frac{k}{2} \right\}.
\end{aligned}$$

Making use of (8) and the monotonicity of the potential $\tilde{V}$ on $(-\infty, 0)$ and on $(0, +\infty)$, we have

$$\begin{aligned}
\int_{\{s\in\mathbb{R}:|Au(s)|>6\pi\}} V(Au(s))ds &\leq \int_{\{s\in\mathbb{R}:|Au(s)|>6\pi\}} \tilde{V}(Au(s))\, ds \\
&\leq \int_{\{s\in\mathbb{R}:|Au(s)|>6\pi\}} \tilde{V}(2\max\{|u(s + \cos\varphi)|, |u(s)|\})\, ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\{s\in\mathbb{R}:\max\{|u(s+\cos\varphi)|,|u(s)|\}>3\pi\}} \frac{1}{4}F(\max\{|u(s+\cos\varphi)|,|u(s)|\})\,ds\\
&\le 2\int_{\{s\in\mathbb{R}:|u(s)|>3\pi\}} \frac{1}{4}F(u(s))ds \le \frac{1}{2}\int_{-\infty}^{+\infty} F(u(s))\,ds. \qquad (11)
\end{aligned}
$$

Similarly,

$$
\int_{\{s\in\mathbb{R}:|Bu(s)|>6\pi\}} V(Bu(s))ds \le \frac{1}{2}\int_{-\infty}^{+\infty} F(u(s))ds. \qquad (12)
$$

By the definition of c_1, we obtain

$$
\begin{aligned}
\int_{\{s\in\mathbb{R}:|Au(s)|\le 6\pi\}} V(Au(s))\,ds &\le \int_{\{s\in\mathbb{R}:|Au(s)|\le 6\pi\}} \frac{c_1^2}{2}(Au(s))^2\,ds\\
&\le \int_{-\infty}^{+\infty} \frac{c_1^2}{2}(Au(s))^2\,ds,\\
\int_{\{s\in\mathbb{R}:|Bu(s)|\le 6\pi\}} V(Bu(s))\,ds &\le \int_{\{s\in\mathbb{R}:|Bu(s)|\le 6\pi\}} \frac{c_1^2}{2}(Bu(s))^2\,ds\\
&\le \int_{-\infty}^{+\infty} \frac{c_1^2}{2}(Bu(s))^2\,ds.
\end{aligned}
$$

Then it follows from (11) and (12) that

$$
\begin{aligned}
\tilde{J}(u) \ge& \int_{-\infty}^{+\infty} \Big[\frac{c^2}{2}(u'(s))^2 - \frac{c_1^2}{2}(Au(s))^2 - \frac{c_1^2}{2}(Bu(s))^2\\
&+K(1+\cos(u(s))) + F(u(s))\Big]\,ds\\
&-\int_{\{s\in\mathbb{R}:|Au(s)|>6\pi\}} V(Au(s))ds - \int_{\{s\in\mathbb{R}:|Bu(s)|>6\pi\}} V(Bu(s))\,ds\\
\ge& \int_{-\infty}^{+\infty} \Big[\frac{c^2-c_1^2}{2}(u'(s))^2 + K(1+\cos(u(s))) + \frac{1}{2}F(u(s))\Big]\,ds
\end{aligned}
$$

for all $u\in E$, and (9) holds true.

Applying Lemma 3.3 to the functional

$$
I_1(u) = \frac{c^2-c_1^2}{2}\int_{-\infty}^{+\infty} [(u'(s))^2 + W_1(u(s))]\,ds,
$$

where

$$
W_1(x) := \frac{2K}{c^2-c_1^2}[1+\cos x + \frac{1}{2K}F(x)],
$$

and making use of (9), we obtain

$$
\begin{aligned}
\inf_{u\in\mathcal{M}_{-\pi,\pi}} \tilde{J}(u) &\ge (c^2-c_1^2)\left|\int_{-\pi}^{\pi}\sqrt{W_1(x)}\,dx\right|\\
&= \sqrt{2(c^2-c_1^2)K}\left|\int_{-\pi}^{\pi}\sqrt{1+\cos x+0}\,dx\right| = 8\sqrt{(c^2-c_1^2)K}.
\end{aligned}
$$

Furthermore, since $V \geq 0$, we have

$$\tilde{J}(u) \leq \frac{c^2}{2}\int_{-\infty}^{+\infty}\left[(u'(s))^2 + \frac{2}{c^2}\left(K(1+\cos(u(s))) + \frac{3}{2}F(u(s))\right)\right] ds.$$

Now, we apply Lemma 3.3 to the functional

$$I_2(u) = \frac{c^2 - c_1^2}{2}\int_{-\infty}^{+\infty}[(u'(s))^2 + W_2(u(s))]\, ds,$$

where

$$W_2(x) := \frac{2K}{c^2}[1 + \cos x + \frac{3}{2K}F(x)].$$

As a consequence, we obtain

$$\inf_{u\in\mathcal{M}_{-\pi,\pi}} \tilde{J}(u) \leq c^2\left|\int_{-\pi}^{\pi}\sqrt{W_2(x)}dx\right| < 8c\sqrt{K},$$

from which inequalities (10) follow. $\square$

The following lemma can be proved in the same way as Lemma 2.7 from [16].

Lemma 3.5. *Assume conditions (i)–(iv) hold. Let $\tilde{u} \in \mathcal{M}_{-\pi,\pi}$ be a minimizer of $\tilde{J}$ on $\mathcal{M}_{-\pi,\pi}$, then*

$$\|\tilde{u}\|_{L^\infty(\mathbb{R})} \leq \frac{3}{2}\pi + \delta,$$

where

$$\delta := \frac{4c_1^2}{c^2 - c_1^2 + c\sqrt{c^2 - c_1^2}}. \tag{13}$$

In particular, if the speed c is large enough to ensure $\delta < \pi$, then $\|\tilde{u}\|_{L^\infty(\mathbb{R})} \leq \frac{5}{2}\pi$.

4. Main result

In order to prove the main result, we need the following version of the concentration-compactness principle obtained in [15, Lemma 4.1] (see [16, 17, 19] for other versions of this principle).

Given $T > 1$ and $\eta \in \mathbb{R}$, we define a truncated version of $\tilde{J}$ by

$$\begin{aligned}\tilde{J}_T(u,\eta) := &\int_0^1\int_{\eta-T+\tau}^{\eta+T-1+\tau}\frac{c^2}{2}(u'(s))^2\, ds\, d\tau - \int_{\eta-T}^{\eta+T-1}V(Au(s))\, ds \\ &- \int_{\eta-T}^{\eta+T-1}V(Bu(s))\, ds + \int_{\eta-T+\frac{1}{2}}^{\eta+T-\frac{1}{2}}\left[K\big(1+\cos(u(s))\big) + \frac{3}{2}F(u(s))\right] ds.\end{aligned}$$

Lemma 4.1 (Concentration-compactness)**.** *Assume conditions (i)–(iv) hold. Let $(u_n) \subset \mathcal{M}_{-\pi,\pi}$ be a minimizing sequence for $\tilde{J}$ on $\mathcal{M}_{-\pi,\pi}$, and let c be large enough to ensure $\delta < \pi$ for δ defined in (13). Then there exists a subsequence, still denoted by (u_n), such that one of the following holds:*

(i$_2$) *(concentration) there is a sequence* $(\eta_n) \subset \mathbb{R}$ *such that for all small enough* $\varepsilon > 0$ *there exists* $T > 0$ *such that*

$$|\tilde{J}(u_n) - \tilde{J}_T(u_n, \eta_n)| < \varepsilon$$

for every $n \in \mathbb{N}$;

(ii$_2$) *(vanishing) for all* $T > 0$,

$$\lim_{n\to\infty} \sup_{\eta\in\mathbb{R}} \tilde{J}_T(u_n, \eta) = 0;$$

(iii$_2$) *(dichotomy) there exists* $\varepsilon_1 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_1$ *there are* $(f_n), (g_n) \subset E$ *such that*

$$|u_n - (f_n + g_n - \pi)| \le \varepsilon, \quad |\tilde{J}(u_n) - (\tilde{J}(f_n) + \tilde{J}(g_n)| \le \varepsilon,$$
$$\lim_{n\to\infty} \operatorname{dist}(\operatorname{supp}(f_n'), \operatorname{supp}(g_n')) = +\infty, \quad \lim_{n\to\infty} \tilde{J}(f_n) = \alpha, \lim_{n\to\infty} \tilde{J}(g_n) = \beta,$$

for some $0 < \alpha, \beta < \inf_{u\in\mathcal{M}_{-\pi,\pi}} \tilde{J}(u)$ *(π is needed in the first inequality to ensure* $J(f_n) < +\infty$ *and* $J(g_n) < +\infty$*).*

Lemma 4.2. *Under the assumptions of Lemma 4.1, the functional* $\tilde{J}$ *has a minimizer on* $\mathcal{M}_{-\pi,\pi}$.

Proof. By Lemma 3.4, the functional $\tilde{J}$ is bounded from below on $\mathcal{M}_{-\pi,\pi}$. Let $(u_n) \subset \mathcal{M}_{-\pi,\pi}$ be a minimizing sequence. Then, by Lemma 4.1, the subsequence exists, still denoted by (u_n), which satisfies either of the following criteria: concentration, vanishing or dichotomy.

Vanishing is impossible (see the proof of Lemma 5.1 in [15]).

We will show that dichotomy is also impossible. Indeed, as $f_n, g_n \in E$ and $\tilde{J}(f_n)$, $\tilde{J}(g_n) < +\infty$, the analogous statement of Lemma 3.1 (with J replaced by $\tilde{J}$) shows that $f_n(\pm\infty), g_n(\pm\infty) \in \{\pm\pi\}$. Since $f_n + g_n - \pi \in \mathcal{M}_{-\pi,\pi}$, then only $f_n(-\infty) = f_n(+\infty)$ or only $g_n(-\infty) = g_n(+\infty)$. In the first case, we set $\tilde{u}_n := g_n$ and in the second case, $\tilde{u}_n := f_n$. Then $(\tilde{u}_n) \subset \mathcal{M}_{-\pi,\pi}$ and, by (iii$_2$), possibly after passing to a subsequence, we have

$$\lim_{n\to\infty} \tilde{J}(\tilde{u}_n) < \inf_{u\in\mathcal{M}_{-\pi,\pi}} \tilde{J}(u) = \lim_{n\to\infty} \tilde{J}(u_n).$$

We obtained a contradiction to the assumption that $(u_n) \subset \mathcal{M}_{-\pi,\pi}$ is a minimizing sequence of $\tilde{J}$.

Thus (i$_2$) holds. Hence, given $\varepsilon > 0$, there exists a sequence $(\eta_n) \subset \mathbb{R}$ and $T_0 > 0$ such that

$$|\tilde{J}(u_n) - \tilde{J}_{T_0}(u_n, \eta_n)| < \varepsilon.$$

Let $w_n(s) = u_n(\eta_n + s)$. The sequence (w_n) is bounded in E. Indeed, by (9),

$$\|w_n'\|_{L^2(\mathbb{R})} = \|u_n'\|_{L^2(\mathbb{R})} \le \frac{2}{c^2 - c_1^2} J(u_n),$$

and by Lemma 3.5,

$$|w_n(0)| \leq \frac{3}{2}\pi + \delta.$$

Hence, (w_n) contains a subsequence, still denoted by (w_n), that converges weakly to some limit $u \in E$. The convergence is uniform on $[-T_0, T_0]$, and

$$\|u'\|_{L^2(-T_0,T_0)} \leq \lim_{n\to\infty} \inf \|w_n'\|_{L^2(-T_0,T_0)} .$$

Since the functions $V(u)$, $1 + \cos u$ and $F(u)$ belong to $C^1(\mathbb{R})$ and therefore are Lipschitz continuous for $|u| \leq \frac{3}{2}\pi + \delta$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\left| \left(\tilde{J}(u) - \frac{c^2}{2}\|u'\|_{L^2(\mathbb{R})} \right) - \left(\tilde{J}_{T_0}(w_n) - \frac{c^2}{2}\|u'\|_{L^2(-T_0,T_0)} \right) \right| \leq \varepsilon.$$

In fact, this inequality holds for all $T > T_0$ instead of T_0. By Lemma 3.1, $u \in \mathcal{M}_{-\pi,\pi}$. Furthermore, as $T \mapsto \tilde{J}_T(w_n, 0)$ is non-decreasing for every $n \in \mathbb{N}$, we obtain that $\tilde{J}_T(w_n, 0) \leq \tilde{J}(w_n)$. Then,

$$\begin{aligned}\tilde{J}(u) = \lim_{T\to\infty} \tilde{J}_T(u, 0) &\leq \lim_{T\to\infty} \lim_{n\to\infty} \inf \tilde{J}_T(w_n, 0)\\ &\leq \lim_{T\to\infty} \lim_{n\to\infty} \tilde{J}(w_n) = \lim_{n\to\infty} \tilde{J}(w_n) = \lim_{n\to\infty} \tilde{J}(u_n),\end{aligned}$$

and thus u is a minimizer of the functional $\tilde{J}$ on $\mathcal{M}_{-\pi,\pi}$. □

The following theorem is the main result of the paper.

Theorem 4.3. *Assume conditions (i)–(iv) hold. Suppose that c is large enough to ensure $\delta < \pi$ for δ defined by (13). Then equation (2) has a solution u that satisfies boundary conditions (3).*

Proof. By Lemma 3.1, the modified functional $\tilde{J}$ has a minimizer $u_* \in \mathcal{M}_{-\pi,\pi}$. We have to show that u_* is a solution of equation (2) with boundary conditions (3). We define the functional $\tilde{\Psi}$ similarly to Ψ but in terms of $\tilde{J}$. Then the function $v_* = u_* - v_0$ minimizes $\tilde{\Psi}$ on $H^1(\mathbb{R})$. Since the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ is continuous, we have that

$$\|v_0 + v\|_{L^\infty(\mathbb{R})} < \frac{5}{2}\pi$$

for all v in the neighborhood $\Delta \subset H^1(\mathbb{R})$ of v_*. Then, by Remark 3.2, for all $v \in \Delta$,

$$\Psi(v) = J(v_0 + v) = \tilde{J}(v_0 + v) = \tilde{\Psi}(v),$$

and v_* minimizes Ψ as well as $\tilde{\Psi}$ in Δ. In particular, v_* is a local minimizer of the functional Ψ on $H^1(\mathbb{R})$, i.e., v_* is a critical point of Ψ. Hence, by Lemma 3.1 (iv$_1$), $u_* = v_0 + v_*$ is the solution of equation (2) that satisfies boundary conditions (3). □

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S. Bak,
Vinnytsia Mykhailo Kotsiubynskyi State Pedagogical University, 32 Ostrozkogo St., Vinnytsia, 21001, Ukraine,
E-mail: sergiy.bak@gmail.com

Існування гетероклінічних рухомих хвиль в дискретному рівнянні синус-Ґордона на двовимірній ґратці

С. Бак

Статтю присвячено дискретному рівнянню синус-Ґордона, яке описує нескінченну систему нелінійно зв'язаних нелінійних осциляторів на двовимірній ґратці із зовнішнім потенціалом $V(r) = K(1 - \cos r)$. Основний результат стосується існування розв'язків у вигляді гетероклінічних рухомих хвиль. За допомогою методу критичних точок і принципу концентрованої компактності отримано достатні умови існування таких розв'язків.

Ключові слова: дискретне рівняння синус-Ґордона, нелінійні осцилятори, двовимірна ґратка, гетероклінічні рухомі хвилі, критичні точки, принцип концентрованої компактності.