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Traveling waves in systems of oscillators on 2D-lattices

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Abstract. A system of differential equations that describes the dynamics of an infinite system of linearly coupled nonlinear oscillators on a 2D-lattice is considered. The exponential estimate of the solution and some results on the existence of periodic and solitary traveling waves are obtained.

Keywords. Oscillators, traveling waves, critical points, mountain pass theorem.

1. Introduction

In the present work, we study the equations describing the dynamics of an infinite system of linearly coupled nonlinear oscillators positioned on a plane integer-valued lattice. Let $q_{n,m}(t)$ be the generalized coordinate of the (n,m)-th oscillator at the time t. It is assumed that each oscillator interacts linearly with its four nearest neighbors. The equations of motion of the system under consideration take the form

$$\ddot{q}_{n,m} = -U'(q_{n,m}) + c_1^2(q_{n+1,m} + q_{n-1,m} - 2q_{n,m}) + c_2^2(q_{n,m+1} + q_{n,m-1} - 2q_{n,m}), \quad (n,m) \in \mathbb{Z}^2. \tag{1.1}$$

Equations (1.1) represent an infinite system of ordinary differential equations.

Similar systems are of interest in connection with numerous physical applications [1, 4, 5]. In works [2, 3, 8], traveling waves in chains of oscillators were studied. The review of the available results concerning such systems is given in [11].

The periodic solutions for a system of oscillators on a two-dimensional lattice were studied in [14], and the traveling waves in similar systems of somewhat different types were considered in [6] and [7] within other methods. In particular, the system with odd 2π-periodic nonlinearity was analyzed in [6].

Here, we will study the question about the existence of periodic and solitary traveling waves within the method of critical points and will establish the exponential estimate of the profile of a traveling wave.

2. Statement of the problem

Consider the system of oscillators with the potential

$$U(r) = -\frac{a}{2}r^2 + V(r).$$

Then the equation of motion takes the form

$$\ddot{q}_{n,m} = c_1^2\triangle_{(1)}q_{n,m} + c_2^2\triangle_{(2)}q_{n,m} + aq_{n,m} - V'(q_{n,m}), \tag{2.1}$$

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where

$$(\triangle_{(1)}q)_{n,m} = q_{n+1,m} + q_{n-1,m} - 2q_{n,m}$$

and

$$(\triangle_{(2)}q)_{n,m} = q_{n,m+1} + q_{n,m-1} - 2q_{n,m}$$

are discrete Laplace operators, respectively, with respect to the variables n and m, and $c_1^2 > 0, c_2^2 > 0$. If $c_1^2 = c_2^2 = 1$, then the linear operator on the right-hand side of (2.1) is a two-dimensional Laplace discrete operator

$$(\triangle q)_{n,m} = q_{n+1,m} + q_{n-1,m} + q_{n,m+1} + q_{n,m-1} - 4q_{n,m}.$$

A traveling wave has the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct),$$

and its profile $u(s)$, where $s = n\cos\varphi + m\sin\varphi - ct$, satisfies the equation

$$c^2u''(s) = c_1^2(u(s+\cos\varphi) + u(s-\cos\varphi) - 2u(s)) \\ + c_2^2(u(s+\sin\varphi) + u(s-\sin\varphi) - 2u(s)) + au(s) - V'(u(s)). \quad (2.2)$$

We note that the function of a continuous argument $u(s)$, $s \in \mathbb{R}$, is called the profile of a wave. The constant $c \neq 0$ is the velocity of the wave. If $c > 0$, then the wave propagates to the right, and if $c < 0$, it moves to the left. Of interest are the nontrivial waves, whose profiles are not identically equal to zero.

The important role is played by the quantity $c_0(\varphi)$ defined by the equation

$$c_0^2(\varphi) = c_1^2\cos^2\varphi + c_2^2\sin^2\varphi. \quad (2.3)$$

In the case of periodic traveling waves, the profile of a wave can be determined by solving Eq. (2.2) with the condition of periodicity

$$u(s + 2k) = u(s), s \in \mathbb{R}. \quad (2.4)$$

The profile of a solitary wave is given by the solution of Eq. (2.2) with the boundary condition at infinity

$$\lim_{s\to\pm\infty} u(s) = u(\pm\infty) = 0. \quad (2.5)$$

We note that, in the case where $\varphi \equiv 0, \pi/2 \mod \pi$, the wave propagates along the appropriate coordinate axis. Such waves are reduced to those on a one-dimensional lattice that were studied in [2,3]. Thus, the results of the present work contain those in [2,3] as particular cases.

Everywhere below, the solution of Eq. (2.2) means a function $u(s)$ of the class $C^2(\mathbb{R})$ satisfying Eq. (2.2) for all $s \in \mathbb{R}$.

3. Variational statement of the problem

Everywhere below, we assume that the potential $V(r)$ satisfies the condition

(*h*) *the function* $V(r)$ *is continuously differentiable,* $V(0) = 0$, $V'(r) = o(r)$ *as* $r \to 0$, *and there exists* $\mu > 2$ *such that*

$$0 < \mu V(r) \leq V'(r)r, r \neq 0.$$

We note that Eq. (2.2) includes only the square of the velocity c. This implies that if the function $u(s)$ satisfies Eq. (2.2), then there exist two traveling waves with the given profile and the velocities $\pm c$. One wave moves to the right, and another one does to the left.

Equation (2.2) and condition (2.4) are related to a functional J_k defined on the space

$$E_k = \{u \in H^1_{loc}(\mathbb{R}) : u(s+2k) = u(s)\}$$

with the norm

$$\|u\|_k = (\|u\|^2_{L^2(-k,k)} + \|u'\|^2_{L^2(-k,k)})^{1/2} = \left(\int_{-k}^{k} (u(s)^2 + u'(s)^2)\, ds \right)^{1/2}.$$

Thus, E_k is the Sobolev space of $2k$-periodic functions. The functional is defined by the equality

$$J_k(u) = \int_{-k}^{k} \Big\{ \frac{c^2}{2}(u'(s))^2 - \frac{c_1^2}{2}(u(s+\cos\varphi) - u(s))^2 - \frac{c_2^2}{2}(u(s+\sin\varphi) - u(s))^2 + \frac{a}{2}u^2(s) - V(u(s)) \Big\}\, ds. \quad (3.1)$$

Problem (2.2), (2.5) is related to the functional

$$J(u) = \int_{-\infty}^{\infty} \Big\{ \frac{c^2}{2}(u'(s))^2 - \frac{c_1^2}{2}(u(s+\cos\varphi) - u(s))^2 - \frac{c_2^2}{2}(u(s+\sin\varphi) - u(s))^2 + \frac{a}{2}u^2(s) - V(u(s)) \Big\}\, ds, \quad (3.2)$$

that is defined on the space $E = H^1(\mathbb{R})$ with the standard Sobolev norm.

We recall that, by the embedding theorem, $E_k \subset C([-k,k])$ and $E \subset C_b(\mathbb{R})$, where $C([-k,k])$ and $C_b(\mathbb{R})$ is the space of continuous functions on $[-k,k]$ and the space of bounded continuous functions on $\mathbb{R}$, respectively. Moreover, the functions from E have zero limit at infinity.

In what follows, we will need

Lemma 3.1. *The inequalities*

$$\|u(\cdot+\alpha) - u(\cdot)\|_{L^2(-k,k)} \le |\alpha|\|u'\|_{L^2(-k,k)}, \quad u \in E_k \quad (3.3)$$

for any $\alpha \in (-k,k)$ and

$$\|u(\cdot+\alpha) - u(\cdot)\|_{L^2(\mathbb{R})} \le |\alpha|\|u'\|_{L^2(\mathbb{R})}, \quad u \in E \quad (3.4)$$

for any $\alpha \in \mathbb{R}$ are valid.

Proof. Let $v_\alpha = u(s+\alpha) - u(s)$, and let

$$\widehat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi t} u(t)\, dt$$

be the Fourier transform of the function u. Then

$$\widehat{v_\alpha}(\xi) = (e^{i\alpha\xi} - 1)\widehat{u}(\xi).$$

We have

$$|\widehat{v_\alpha}(\xi)|^2 = 2(1 - \cos(\alpha\xi))|\widehat{u}(\xi)|^2 = 4\sin^2\frac{\alpha\xi}{2}|\widehat{u}(\xi)|^2 \le \alpha^2\xi^2|\widehat{u}(\xi)|^2.$$

Inequality (3.4) follows now from the Parseval identity.

Inequality (3.3) can be proved similarly with the help of Fourier series. □

Remark 3.1. It follows from the proof that the constant $|\alpha|$ in inequality (3.4) cannot be decreased. For every fixed k, the constant $|\alpha|$ in (3.3) can be decreased. However, it is the least constant, at which inequality (3.3) is valid for all k.

Lemma 3.2. *Under the above assumptions, J_k and J are functionals of the class C^1 on E_k and E, respectively. Their derivatives are given by the formulas*

$$\begin{aligned}\langle J_k'(u), h\rangle = \int_{-k}^{k} \{c^2 u'(s)h'(s) + c_1^2(u(s+\cos\varphi) \\ + u(s-\cos\varphi) - 2u(s))h(s) + c_2^2(u(s+\sin\varphi) + u(s-\sin\varphi) \\ - 2u(s))h(s) + au(s)h(s) - V'(u(s))h(s)\}\, ds \end{aligned} \quad (3.5)$$

and

$$\begin{aligned}\langle J'(u), h\rangle = \int_{-\infty}^{\infty} \{c^2 u'(s)h'(s) + c_1^2(u(s+\cos\varphi) \\ + u(s-\cos\varphi) - 2u(s))h(s) + c_2^2(u(s+\sin\varphi) + u(s-\sin\varphi) \\ - 2u(s))h(s) + au(s)h(s) - V'(u(s))h(s)\}\, ds \end{aligned} \quad (3.6)$$

for $u, h \in E_k$ and $u, h \in E$, respectively.

Proof. By virtue of Lemma 3.1, the quadratic part of the functional J_k is a continuous quadratic functional on E_k and, hence, belongs to the class C^1.

Consider the nonquadratic part

$$\Psi_k(u) = \int_{-k}^{k} V(u(s))\, ds.$$

It is sufficient to demonstrate that Ψ_k belongs to the class C^1 on every open ball of the space E_k with the center at zero. Let B_{r_0} be such a ball with radius $r_0 > 0$. By the embedding theorem, $B_{r_0} \subset \tilde{B}_{r_1}$, where $\tilde{B}_{r_1}$ is an open ball of some radius r_1 in the space $C([-k, k])$. We now fix arbitrarily a continuously differentiable function $\tilde{V}(r)$ such that $\tilde{V}(r) = r$ at $|r| \ge r_2$, where $r_2 > r_1$ is sufficiently large.

Consider the functional

$$\tilde{\Psi}_k(u) = \int_{-k}^{k} \tilde{V}(u(s))\, ds.$$

By construction, $\tilde{\Psi}_k$ coincides with Ψ_k on the ball B_{r_0}. By virtue of the classical results in [9,15], $\tilde{\Psi}_k$ is a C^1 functional on $L^2(-k,k)$ and, hence, on the space E_k continuously embedded in it. This implies that $\tilde{\Psi}_k$ belongs to the class C^1 on B_{r_0} and, by virtue of the arbitrariness of r_0, on the whole E_k.

We can similarly prove that the functional J belongs to the class C^1.

Formulas (3.5) and (3.6) for the derivatives can be obtained by the direct calculation. □

Lemma 3.3. *Critical points of the functionals* J_k *and* J *are* C^2*-solutions of Eq.* (2.2) *that satisfy conditions* (2.4) *and* (2.5), *respectively.*

Proof. Let us consider the case of the functional J (the second case is similar). By the embedding theorem, every element $u \in E$ satisfies condition (2.5). Therefore, it is sufficient only to verify that critical points of J are C^2-solutions of (2.2).

Let $u \in E$ be a critical point of the functional J. Then $\langle J(u), h\rangle = 0$ for any $h \in E$. We choose $h = \varphi \in C_0^\infty(\mathbb{R})$ arbitrarily and use formula (3.6). We have

$$\int_{-\infty}^{\infty} \{c^2 u'(s)h'(s) + c_1^2(u(s+\cos\varphi) + u(s-\cos\varphi) - 2u(s))h(s) + c_2^2(u(s+\sin\varphi) + u(s-\sin\varphi) - 2u(s))h(s) + au(s)h(s) - V'(u(s))h(s)\}\, ds = 0.$$

This means that u satisfies Eq. (2.2) in the sense of distributions. By the embedding theorem, $u \in C_b(\mathbb{R})$. Hence, the right-hand side of (2.2) is a continuous function. So, we conclude that u'' is a continuous function, and, hence, $u \in C^2$ is the solution of Eq. (2.2) in the classical sense. □

4. Main results

We need

Lemma 4.1. *Let condition* (h) *be satisfied, let* $a > 0$, *and let* $c^2 > c_0^2(\varphi)$. *Then there exist* $\varepsilon_0 > 0$ *and* $\gamma > 0$ *that are independent of* $k \geq 1$ *and such that the inequalities*

$$\varepsilon_0 \leq \|u\|_k^2 \leq \gamma J_k(u), \tag{4.1}$$

$$\varepsilon_0 \leq \|u\|^2 \leq \gamma J(u) \tag{4.2}$$

are valid for nontrivial critical points of the functionals J_k *and* J.

Proof. Let $u \in E_k$ be a critical point of the functional J_k. Then $J_k'(u) = 0$, and

$$J_k(u) = J_k(u) - \frac{1}{\mu}\langle J_k'(u), u\rangle = \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_{-k}^{k} \{c^2|u'(s)|^2 - c_1^2|u(s+\cos\varphi) - u(s)|^2 - c_2^2|u(s+\sin\varphi) - u(s)|^2 + a|u(s)|^2\}\, ds - \int_{-k}^{k} \Big\{V(u(s)) - \frac{1}{\mu}V'(u(s))u(s)\Big\}\, ds$$

$$\geq \frac{\mu-2}{2\mu}\Big\{c^2\int_{-k}^{k} |u'(s)|^2\, ds - c_1^2\int_{-k}^{k} |u(s+\cos\varphi) - u(s)|^2\, ds$$

$$- c_2^2 \int_{-k}^{k} |u(s+\sin\varphi) - u(s)|^2 \, ds + a \int_{-k}^{k} |u(s)|^2 \, ds \Bigg\}.$$

Using Lemma 3.1, we obtain

$$J_k(u) \geq \frac{\mu-2}{2\mu} \left\{ \alpha_0 \int_{-k}^{k} |u'(s)|^2 \, ds + a \int_{-k}^{k} |u(s)|^2 \, ds \right\},$$

where $\alpha_0 = c^2 - c_0^2(\varphi)$. Then

$$J_k(u) \geq \frac{\mu-2}{2\mu} \alpha_1 \left\{ \int_{-k}^{k} |u'(s)|^2 \, ds + \int_{-k}^{k} |u(s)|^2 \, ds \right\} = \frac{\mu-2}{2\mu} \alpha_1 \|u\|_k^2,$$

where $\alpha_1 = \min\{\alpha_0, a\}$. This yields the second inequality (4.1).

We now prove the first inequality in (4.1). For a critical point $u \in E_k$, we have $\langle J_k'(u), u \rangle = 0$, i.e.,

$$\int_{-k}^{k} \{c^2 |u'(s)|^2 - c_1^2 |u(s+\cos\varphi) - u(s)|^2 - c_2^2 |u(s+\sin\varphi) - u(s)|^2 + a|u(s)|^2\} \, ds = \int_{-k}^{k} V'(u(s)) \, ds.$$

Hence, as above, we have

$$\alpha_1 \|u\|_k^2 \leq \int_{-k}^{k} V'(u(s)) u(s) \, ds. \tag{4.3}$$

Condition (h) yields

$$V'(r) r \leq \sigma(|r|) r^2,$$

where $\sigma(r)$ is a monotonically increasing continuous function of $r \geq 0$, and $\sigma(0) = 0$. Then (4.3) yields

$$\alpha_1 \|u\|_k^2 \leq \sigma(\|u\|_{C([-k,k])}) \int_{-k}^{k} |u(s)|^2 \, ds.$$

By the embedding theorem,

$$\|u\|_{C([-k,k])} \leq C \cdot \|u\|_k$$

with a constant C independent of k. Hence,

$$\alpha_1 \|u\|_k^2 \leq \sigma(C \cdot \|u\|_k) \|u\|_k^2.$$

Since $u \neq 0$, we have

$$\sigma(C \cdot \|u\|_k) \geq \alpha_1.$$

This yields the first inequality in (4.1) with

$$\varepsilon_0^{1/2} = C^{-1} \cdot \sigma^{-1}(\alpha_1).$$

Inequality (4.2) can be proved similarly with the same constants ε_0 and γ. □

4.1. Existence of periodic traveling waves

With the help of the mountain pass theorem, we will prove the existence of nontrivial traveling waves with periodic profile. By virtue of Lemma 3.3, it is sufficient to establish the existence of nontrivial critical points of the functional J_k. We note that $u = 0$ is always a trivial critical point and gives a trivial traveling wave equal to zero.

Theorem 4.1. *Let condition (h) be satisfied, and let $a > 0$. Then, for any $k \geq 1$ and $c^2 > c_0^2(\varphi)$, Eq. (2.2) has a solution u satisfying condition (2.4). Therefore, there exist two traveling waves with a profile u and velocities $\pm c$. Moreover, there exist constants $\varepsilon > 0$ and $C > 0$ that are independent of k and such that*

$$\varepsilon_0 \leq \|u\|_k^2 \leq C, \tag{4.4}$$

$$\varepsilon_0 \leq J_k \leq C. \tag{4.5}$$

We now formulate the mountain pass theorem in the required form and verify its conditions for the functional J_k ([12, 16]).

Theorem 4.2 (on a mountain pass). *Let I be a functional of the class C^1 on a Hilbert space H that satisfies the Palais–Smale condition:*

(PS) *if a sequence $u_n \in H$ is such that $I'(u_n) \to 0$, and $I(u_n)$ is bounded, then it contains a convergent subsequence.*

We assume that there exist $e \in H$ and $r > 0$ such that $\|e\| > r$ and

$$\beta := \inf_{\|v\|=r} I(v) > 0 = I(0) \geq I(e)\,.$$

Then there exists a critical point $u \in H$ of the functional I such that $I(u) \geq \beta$. In this case,

$$I(u) \leq \sup_{\tau \geq 0} I(\tau e)\,.$$

Let us start with the Palais–Smale condition.

Lemma 4.2. *Under conditions of Theorem 4.1, the functional J_k satisfies the Palais–Smale condition.*

Proof. Let $u_m \in E_k$ be a sequence such that $J_k'(u_m) \to 0$ and $J_k(u_m) \leq C$. Then, like in the beginning of the proof of Lemma 4.1, we have

$$\begin{aligned}
J_k(u_m) - \frac{1}{\mu}\langle J_k'(u_m), u_m\rangle &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int\limits_{-k}^{k} \{c^2 u_m'(s)^2 - c_1^2 |u(s+\cos\varphi) - u(s)|^2 \\
&\quad - c_2^2 |u(s+\sin\varphi) - u(s)|^2 + a u_m(s)^2\}\, ds - \int\limits_{-k}^{k} \left\{V(u_m(s)) - \frac{1}{\mu} V'(u_m(s)) u_m(s)\right\} ds \\
&\geq \frac{\mu - 2}{2\mu} \Bigg\{ c^2 \int\limits_{-k}^{k} u_m'(s)^2\, ds - c_1^2 \int\limits_{-k}^{k} |u(s+\cos\varphi) - u(s)|^2\, ds \\
&\qquad - c_2^2 \int\limits_{-k}^{k} |u(s+\sin\varphi) - u(s)|^2\, ds + a \int\limits_{-k}^{k} u_m(s)^2\, ds \Bigg\}.
\end{aligned}$$

Like in the proof of Lemma 4.1, the lower bound of the right-hand side of this inequality is as follows:

$$\frac{\mu-2}{2\mu}\alpha_1\|u_m\|_k^2.$$

On the other hand, its left-hand side does not exceed

$$C+\frac{1}{\mu}C_1\|u_m\|_k.$$

Hence,

$$\frac{\mu-2}{2\mu}\alpha_1\|u_m\|_k^2 \le C+\frac{1}{\mu}C_1\|u_m\|_k,$$

which yields the boundedness of the sequence u_m in the space E_k.

Since the space E_k is a Hilbert one, we may assume that $u_m \to u$ weakly in E_k. According to the compactness of the embedding $E_k \subset C([-k,k])$, the last convergence is strong in $C([-k,k])$.

For the sake of brevity, we set $u_{m,l} = u_m - u_l$. According to Lemma 3.2, we have

$$\begin{aligned}\langle J_k'(u_m) - J_k'(u_l), u_{m,l}\rangle = \int_{-k}^{k} \{c^2(u_{m,l}(s))^2 - c_1^2(u_{m,l}(s+\cos\varphi) - u_{m,l}(s))^2 \\ - c_2^2(u_{m,l}(s+\sin\varphi) - u_{m,l}(s))^2 + a(u_{m,l}(s))^2\}\,ds \\ - \int_{-k}^{k} \{V'(u_m(s)) - V'(u_l(s))\}u_{m,l}(s)\,ds.\end{aligned}$$

Hence, with the help of the same trick, like that in the proof of Lemma 4.1, we obtain

$$\langle J_k'(u_m) - J_k'(u_l), u_{m,l}\rangle \ge \alpha_1\|u_{m,l}\|_k^2 - \int_{-k}^{k} \{V'(u_m(s)) - V'(u_l(s))\}u_{m,l}\,ds,$$

or

$$\alpha_1\|u_{m,l}\|_k^2 \le \langle J_k'(u_m) - J_k'(u_l), u_{m,l}\rangle + \int_{-k}^{k} \{V'(u_m(s)) - V'(u_l(s))\}u_{m,l}\,ds. \tag{4.6}$$

Since $u_{m,l} \to 0$ weakly in E_k, and $J_k'(u_m) \to 0$ strongly in the dual space E_k^*, the first term on the right-hand side of (4.6) converges to zero as $m, l \to \infty$. In addition, $u_m \to u$ in $C([-k,k])$. This implies that the integrand in (4.6) converges to zero uniformly on $[-k,k]$ as $m, l \to \infty$. Hence, the integral term in (4.6) converges to zero as well. This implies that u_m is a Cauchy sequence in E_k, and, hence, $u_m \to u$ strongly in E_k. □

Lemma 4.3. *Under conditions of Theorem* 4.1, *there exist* $r_0 > 0$ *and* $\alpha_0 > 0$ *that are independent of* k *and such that*

$$\inf_{\|u\|_k = r_0} J_k(u) > \alpha_0.$$

Proof. According to condition (h),

$$V(r) \le \mu^{-1}\sigma(|r|)r^2.$$

Hence, we have

$$J_k(u) = \frac{1}{2}\int_{-k}^{k} \{c^2 u'(s)^2 - c_1^2|u(s+\cos\varphi) - u(s)|^2$$

$$- c_2^2|u(s+\sin\varphi) - u(s)|^2 + au(s)^2\}\, ds - \int_{-k}^{k} V(u(s))\, ds$$

$$\ge \frac{\alpha_1}{2}\|u\|_k^2 - \frac{1}{\mu}\int_{-k}^{k} \sigma(u(s))u(s)^2\, ds \ge \frac{\alpha_1}{2}\|u\|_k^2 - \frac{1}{\mu}\sigma(\|u\|_{C([-k,k])})\|u\|^2_{L^2(-k,k)}$$

$$\ge \frac{\alpha_1}{2}\|u\|_k^2 - \frac{1}{\mu}\sigma(\|u\|_{C([-k,k])})\|u\|_k^2.$$

By the embedding theorem,

$$\|u\|_{C([-k,k])} \le C\|u\|_k.$$

Therefore,

$$J_k(u) \ge \left\{\frac{\alpha_1}{2} - \frac{1}{\mu}\sigma(C\|u\|_k)\right\}\|u\|_k^2.$$

We choose $r_0 > 0$ such that

$$\frac{1}{\mu}\sigma(Cr_0) = \frac{\alpha_1}{4}.$$

This is obviously possible by virtue of properties of the function $\sigma(r)$. Then, at $\|u\|_k = r_0$, we have

$$J_k(u) \ge \frac{\alpha_1 r_0^2}{4},$$

which proves the lemma. □

Let us fix an arbitrary infinitely differentiable function $g \ne 0$ on $\mathbb{R}$ with support in the interval $[0,1]$. Let now v_k be a $2k$-periodic function such that $v_k|_{[-k,k]} = g|_{[-k,k]}$. It is obvious that $v_k \in E_k$.

Lemma 4.4. *Under conditions of Theorem* 4.1, *there exists $\tau_0 > 0$ that is independent of k and such that*

$$J_k(\tau v_k) = J_1(\tau v_1) \le 0$$

for all $\tau \ge \tau_0$.

Proof. By the definition of v_k, we have

$$J_k(\tau v_k) = \frac{1}{2}\int_{-k}^{k} \{c^2\tau^2(g'(s))^2 - \tau^2 c_1^2|g(s+\cos\varphi) - g(s)|^2 - \tau^2 c_2^2|g(s+\sin\varphi) - g(s)|^2 + a\tau^2(g(s))^2\}\, ds$$

$$-\int_{-k}^{k} V(\tau g(s))\,ds = \frac{\tau^2}{2}\int_{-1}^{1}\{c^2(g'(s))^2 - c_1^2|g(s+\cos\varphi) - g(s)|^2 - c_2^2|g(s+\sin\varphi) - g(s)|^2 + a(g(s))^2\}\,ds$$
$$-\int_{0}^{1} V(\tau g(s))\,ds$$

at $k \geq 1$. Condition (h) yields

$$V(\tau g(s)) \geq d\tau^\mu |g(s)|^\mu - d_0.$$

Therefore,

$$J_k(\tau v_k) = J_1(\tau v_1) \leq \gamma_1\tau^2 - d\gamma_2\tau^\mu - d_0,$$

where

$$\gamma_1 = \frac{1}{2}\int_{-1}^{1}\{c^2(g'(s))^2 - c_1^2|g(s+\cos\varphi) - g(s)|^2 - c_2^2|g(s+\sin\varphi) - g(s)|^2 + a(g(s))^2\}\,ds > 0,$$

$$\gamma_2 = \int_{-1}^{1} |g(s)|^\mu\,ds > 0.$$

Since $\mu > 2$, this yields the assertion of the lemma. □

Proof of Theorem 4.1. Lemmas 4.2–4.4 show that the functional J_k satisfies all conditions of the mountain pass theorem. Hence, J_k has a nonzero critical point $u \in E_k$. By Lemma 3.3, u is a C^2-solution of problem (2.2), (2.4). The lower bounds for $\|u\|_k$ and $J_k(u)$ follow from Lemma 4.1. By virtue of Lemma 4.4,

$$J_k(u) \leq \sup_{\tau\geq 0} J_k(\tau v_k) = \sup_{\tau\geq 0} J_1(\tau v_1) = C,$$

and the upper bound for $\|u\|_k$ follows from Lemma 4.1. The theorem is proved. □

4.2. Existence of solitary traveling waves

We now prove the existence of solitary traveling waves. In this case, the traveling waves are determined as critical points of the functional J that satisfies the propositions analogous to Lemmas 4.3 and 4.4. Thus, the functional J satisfies a part of conditions of the mountain pass theorem. However, the Palais–Smale condition for this functional is not satisfied. Therefore, in this case, critical points will be constructed in another way, by considering critical points of the functional J_k and then passing to the limit as $k \to \infty$.

Theorem 4.3. *Let condition* (h) *be satisfied, and let* $a > 0$. *Then, for any* $c^2 > c_0^2(\varphi)$, *Eq.* (2.2) *has a solution* $u \in E$ *satisfying, hence, condition* (2.5). *Thus, there exist two solitary traveling waves with a profile* u *and velocities* $\pm c$.

To prove the theorem, we need the following particular case of Lemma 4.1 from [10].

Lemma 4.5. *Let $u_n \in E_{k_n}$, where $k_n \to \infty$, and let $\|u_n\|_{k_n}$ be bounded. If, for some $r > 0$,*

$$\sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} |u_n(s)|^2\, ds \to 0, \tag{4.7}$$

then $\|u_n\|_{L^p(-k_n,k_n)} \to 0$ for any $p > 2$.

Proof of Theorem 4.3. Let us choose arbitrarily a sequence $k_n \to \infty$. By $u_n \in E_{k_n}$, we denote the solution of Eq. (2.2) under condition (2.4) that is constructed in Theorem 4.1 with $k = k_n$.

Passing to a subsequence, we may assume that there exist $\delta > 0, r > 0$, and a sequence $y_n \in \mathbb{R}$ such that

$$\int_{y_n-r}^{y_n+r} |u_n(s)|^2\, ds \geq \delta. \tag{4.8}$$

Indeed, assume the contrary. Then, for any $r > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} |u_n(s)|^2\, ds = 0.$$

In addition, by virtue of inequality (4.4), the sequence $\|u_n\|_{k_n}$ is bounded. Hence, according to Lemma 4.5, it follows that

$$\|u_n\|_{L^p(-k_n,k_n)} \to 0. \tag{4.9}$$

Further, $J_k'(u_n) = 0$ and, hence, $\langle J_k'(u_n), u_n\rangle = 0$, i.e.,

$$\int_{-k_n}^{k_n} \{c^2(u_n'(s))^2 - c_1^2|u_n(s+\cos\varphi) - u_n(s)|^2 - c_2^2|u_n(s+\sin\varphi) - u_n(s)|^2 + a(u_n(s))^2\}\, ds$$

$$= \int_{-k_n}^{k_n} V'(u_n(s))u_n(s)\, ds.$$

This yields

$$\alpha_1\|u_n\|_{k_n}^2 \leq \int_{-k_n}^{k_n} V'(u_n(s))u_n(s)\, ds. \tag{4.10}$$

By virtue of the embedding theorem, the functions $u_n(s)$ are continuous and uniformly bounded in n, i.e., there exists $R > 0$ such that $|u_n(s)| \leq R$. Let us fix an arbitrary $p > 2$. According to condition (h), for any $\varepsilon > 0$, there exists $C = C_\varepsilon$ such that

$$|V'(r)| \leq \varepsilon|r| + C|r|^{p-1}$$

at $|r| \leq R$. Then inequality (4.10) yields

$$\alpha_1\|u_n\|_{k_n}^2 \leq \varepsilon \int_{-k_n}^{k_n} |u_n(s)|^2\, ds + C \int_{-k_n}^{k_n} |u_n(s)|^p\, ds$$

$$= \varepsilon \|u_n\|^2_{L^2(-k_n,k_n)} + C\|u_n\|^p_{L^p(-k_n,k_n)} \le \varepsilon \|u_n\|^2_{k_n} + C\|u_n\|^p_{(-k_n,k_n)}.$$

Choosing $\varepsilon = \alpha_1/2$, we obtain

$$\frac{\alpha_1}{2}\|u_n\|^2_{k_n} \le C\|u_n\|^p_{L^p(-k_n,k_n)}.$$

Then, according to (4.9), $\|u_n\|_{k_n} \to 0$, which contradicts the first inequality in (4.4). Thus, (4.8) is proved.

Equation (2.2) is translation invariant. Therefore, if $u(s)$ is its solution, then $u(s+y)$ is also a solution for any $y \in \mathbb{R}$. Hence, replacing $u_n(s)$ by $u_n(s+y_n)$, we may assume that (4.8) is satisfied with $y_n = 0$.

Since $\|u_n\|_{k_n}$ is bounded, we may assume, by passing to a subsequence, that $u_n \to u$ weakly in $H^1_{loc}(\mathbb{R})$, i.e., weakly in $H^1(a,b)$ for any finite interval (a,b). According to the embedding theorem, $u_n \to u$ uniformly on any finite interval. Therefore, we can pass to the limit in inequality (4.8) (with $y_n = 0$) and obtain

$$\int_{-r}^{r} |u(s)|^2\, ds \ge \delta.$$

This shows that $u \ne 0$.

We now prove that $u \in E$. Let us choose arbitrarily $b > 0$. For sufficiently large n, we have

$$\int_{-b}^{b} \{|u_n'(s)|^2 + |u_n(s)|^2\}\, ds \le \int_{-k_n}^{k_n} \{|u_n'(s)|^2 + |u_n(s)|^2\}\, ds \le C,$$

by virtue of the boundedness of $\|u_n\|_{k_n}$. Since $u_n \to u$ weakly in $H^1(-b,b)$, we have

$$\int_{-b}^{b} \{|u'(s)|^2 + |u(s)|^2\}\, ds \le \lim_{n\to\infty} \inf \int_{-b}^{b} \{|u_n'(s)|^2 + |u_n(s)|^2\}\, ds \le C.$$

Since b is arbitrary, this yields

$$\|u\|^2 = \int_{-\infty}^{\infty} \{|u'(s)|^2 + |u(s)|^2\}\, ds \le C < \infty,$$

i.e., $u \in E$.

It remains to verify that u is a solution of Eq. (2.2). Let $g(s)$ be any infinitely differentiable function with a compact support $\operatorname{supp} g(s) \subset [-b,b]$. For sufficiently large n, the interval $(-k_n+1, k_n-1)$ contains $[-b,b]$, and, hence, the function $g_n \in E_{k_n}$ that coincides with g on $(-k_n,k_n)$ is well-defined. Since u_n is a critical point of the functional J_k, we have

$$0 = \langle J'_{k_n}(u_n), g_n\rangle = \int_{-k_n}^{k_n} \{c^2 u_n'(s) g_n'(s) - c_1^2(u_n(s+\cos\varphi) + u_n(s-\cos\varphi)$$

$$- 2u_n(s))g_n(s) - c_2^2(u_n(s+\sin\varphi) + u_n(s-\sin\varphi) - 2u_n(s))g_n(s)$$

$$+ au_n(s)g_n(s)\}\,ds - \int_{-k_n}^{k_n} V'(u_n(s))g_n(s)\,ds$$

$$= \int_{-b}^{b} \{c^2 u_n'(s)g'(s) - c_1^2(u_n(s+\cos\varphi) + u_n(s-\cos\varphi) - 2u_n(s))g(s)$$

$$- c_2^2(u_n(s+\sin\varphi) + u_n(s-\sin\varphi) - 2u_n(s))g(s) + au_n(s)g(s)\}\,ds - \int_{-b}^{b} V'(u_n(s))g(s)\,ds.$$

In the first integral on the right-hand side of this equality, we can pass to the limit as $n \to \infty$, since $u_n \to u$ weakly in $H^1(-b, b)$. According to the embedding theorem, $u_n \to u$ uniformly on $[-b, b]$. Therefore, we can pass to the limit in the second integral as well. Thus,

$$0 = \int_{-b}^{b} \{c^2 u'(s)g'(s) - c_1^2(u_n(s+\cos\varphi) + u_n(s-\cos\varphi)$$

$$- 2u_n(s))g(s) - c_2^2(u_n(s+\sin\varphi) + u_n(s-\sin\varphi) - 2u_n(s))g(s)$$

$$+ au(s)g(s)\}\,ds - \int_{-b}^{b} V'(u(s))g(s)\,ds$$

$$= \int_{-\infty}^{\infty} \{c^2 u'(s)g'(s) - c_1^2(u_n(s+\cos\varphi) + u_n(s-\cos\varphi)$$

$$- 2u_n(s))g(s) - c_2^2(u_n(s+\sin\varphi) + u_n(s-\sin\varphi) - 2u_n(s))g(s)$$

$$+ au(s)g(s) - V'(u(s))g(s)\}\,ds = \langle J'(u), g\rangle.$$

Since g is any infinitely differentiable function with a compact support, and the set of such functions is dense in E, we have $J'(u) = 0$. This means that u is a critical point of the functional J and, hence, a solution of the problem under consideration. The theorem is proved. □

4.3. Exponential decay of the profile of a solitary wave

Equation (2.2) can be written in the form

$$Lu = f(u), \tag{4.11}$$

where

$$Lu(t) = -c^2 u''(t) + c_1^2(u(t+\cos\varphi) + u(t-\cos\varphi) - 2u(t)) + c_2^2(u(t+\sin\varphi) + u(t-\sin\varphi) - 2u(t)) + au(t) \tag{4.12}$$

and $f(r) = V'(r)$. As for the function $f(r)$, we made the following assumption weaker than (h):

(h') *$f(r)$ is continuous on $\mathbb{R}$, $f(0) = 0$, $f(r) = o(r)$ as $r \to 0$, and $f(r) = 0$ at $r \neq 0$.*

We now consider solutions lying in the space $E = H^1(\mathbb{R})$. Let $u \in E$ be such a solution. We set

$$g(t) = \frac{f(u(t))}{u(t)}$$

(if $u(t) = 0$, then $g(t) = 0$ by definition). Condition (h') yields

$$\lim_{t \to \pm\infty} g(t) = 0.$$

Equation (4.11) takes the form

$$Lu(t) = g(t) \cdot u(t). \tag{4.13}$$

By applying the Fourier transformation to Eq. (3.1), we obtain

$$\sigma(\xi)\widehat{u}(\xi) = \widehat{g \cdot u}(\xi), \tag{4.14}$$

where

$$\sigma(\xi) = c^2\xi^2 - 4c_1^2 \sin^2\left(\frac{\xi}{2}\cos\varphi\right) - 4c_2^2 \sin^2\left(\frac{\xi}{2}\sin\varphi\right) + a. \tag{4.15}$$

We note that the function $\sigma(\xi), \xi \in \mathbb{R}$, extends to the entire function

$$\sigma(\zeta) = c^2\zeta^2 - 4c_1^2 \sin^2\left(\frac{\zeta}{2}\cos\varphi\right) - 4c_2^2 \sin^2\left(\frac{\zeta}{2}\sin\varphi\right) + a, \zeta \in \mathbb{C}.$$

Lemma 4.6. *Let $c^2 > c_0^2(\varphi)$, and let $a > 0$. Then there exists $\beta_0 > 0$, such that the function $\sigma(\zeta)$ has no zeros in the strip $|\operatorname{Im}\zeta| < \beta_0$.*

Proof. First of all, we note that $\sigma(\xi) > 0$ at all $\xi \in \mathbb{R}$, and, hence, σ does not vanish on the real axis. Indeed, from the inequality

$$|\sin x| \leq |x|,$$

we have

$$\begin{aligned}\sigma(\xi) = c^2 - 4c_1^2 \sin^2\left(\frac{\xi}{2}\cos\varphi\right) - 4c_2^2 \sin^2\left(\frac{\xi}{2}\sin\varphi\right) + a \\ \geq c^2\xi^2 - \xi^2(c_1^2\cos^2\varphi + c_2^2\sin^2\varphi) + a \geq (c^2 - c_0^2(\varphi))\xi^2 + a \geq a > 0.\end{aligned}$$

Let now $A > 0$ be arbitrary, and let $|\operatorname{Im}\zeta| < A$. Writing ζ in the form $\zeta = \zeta + i\tau$, we have $|\tau| < A$ and

$$\begin{aligned}\left|\sin\left(\frac{\zeta}{2}\cos\varphi\right)\right| &= \frac{1}{2}|e^{i\zeta\cos\varphi/2} - e^{-i\zeta\cos\varphi/2}| \\ &= \frac{1}{2}|e^{i\xi\cos\varphi/2}e^{-\tau\cos\varphi/2} - e^{-i\xi\cos\varphi/2}e^{\tau\cos\varphi/2}| \\ &\leq \frac{1}{2}(|e^{i\xi\cos\varphi/2}e^{-\tau\cos\varphi/2}| + |e^{-i\xi\cos\varphi/2}e^{\tau\cos\varphi/2}|) \\ &= \frac{1}{2}(e^{-\tau\cos\varphi/2} + e^{\tau\cos\varphi/2}) \leq e^{A|\cos\varphi|/2}.\end{aligned}$$

Thus,

$$\left|\sin^2\left(\frac{\zeta}{2}\cos\varphi\right)\right| \leq e^{A|\cos\varphi|}.$$

Similarly,

$$\left|\sin^2\left(\frac{\zeta}{2}\sin\varphi\right)\right| \le e^{A|\sin\varphi|}.$$

Then

$$|\sigma(\xi+i\tau)| \ge c^2|\xi+i\tau|^2 - 4c_1^2 e^{A|\cos\varphi|} - 4c_2^2 e^{A|\sin\varphi|} + a.$$

Therefore, if $|\xi|$ is sufficiently large, and if $|\tau| < A$, then $|\sigma(\xi+i\tau)| > 0$, and, hence, $\sigma(\zeta) \ne 0$ for such $\zeta = \xi + i\tau$. Thus, there exists $B > 0$ such that the function $\sigma(\zeta)$ does not vanish whenever $|\tau| < A$, $|\xi| \ge B$. In addition, the analytic function $\sigma(\zeta)$ can have at most a finite number of zeros in the rectangle $|\xi| < B$, $|\tau| < A$. This yields immediately the existence of $\beta_0 > 0$ such that the function $\sigma(\zeta)$ has no zeros in the strip $|\tau| < \beta_0$. ☐

Further, we need the following proposition ([11, Lemma 4.8]).

Lemma 4.7. *Let $f(t)$ and $g(t)$ be bounded nonnegative functions on $\mathbb{R}$, and $\lim_{t\to\pm\infty} g(t) = 0$. Let also*

$$f(t) \le \int_{-\infty}^{+\infty} e^{-\beta|t-s|} g(s) f(s)\, ds,$$

with $\beta > 0$. Then, for any $\alpha \in (0,\beta)$, there exists a constant $C = C(\alpha)$ such that

$$f(t) \le Ce^{-\alpha|t|}.$$

The following theorem is valid:

Theorem 4.4. *Let condition (h') be satisfied, let $c^2 > c_0^2(\varphi)$, and let $a > 0$. If $u \in E$ is a solution of Eq.* (2.2)*, then, for any $\beta \in (0,\beta_0)$, where β_0 is defined in Lemma* 4.6*, there exists $C_\beta > 0$ such that*

$$|u(t)| \le C_\beta e^{-\beta|t|}. \tag{4.16}$$

Proof. Equation (3.2) yields

$$\widehat{u}(\xi) = \frac{1}{\sigma(\xi)}\widehat{g\cdot u}(\xi).$$

Let

$$K(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{it\xi} \cdot \frac{1}{\sigma(\xi)}\, d\xi.$$

Then

$$u(t) = [K * (g\cdot u)](t) = \int_{-\infty}^{+\infty} K(t-s) g(s) u(s)\, ds. \tag{4.17}$$

According to Lemma 4.6, the function $1/\sigma(\zeta)$ is analytic in the strip $|\operatorname{Im}\zeta| < \beta_0$. Therefore, by the Paley–Wiener theorem (see [13, Theorem IX.14]), the estimate

$$|K(t)| \le C_\beta e^{-\beta|t|}$$

is valid for any $\beta \in (0,\beta_0)$. Equation (4.17) yields

$$|u(t)| \le C_\beta \int_{-\infty}^{+\infty} e^{-\beta|t-s|} |g(s)||u(s)|\, ds.$$

Now, by virtue of Lemma 4.7, we obtain Q.E.D. ☐

Since condition (h) is stronger than condition (h'), Theorem 4.4 yields

Corollary 1. *Under conditions of Theorem* 4.4, *the solution* u *satisfies the exponential estimate* (4.16) *for any* $\beta \in (0, \beta_0)$.

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